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Yang–Lee Zeros of the Potts Model

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The Yang–Lee zeros of the three-component ferromagnetic Potts model in one dimension in the complex plane of an applied field are determined. The phase diagram consists of a triple point where three phases coexist. Emerging from the triple point are three lines on which two phases coexist and which terminate at critical points (Yang–Lee edge singularity). The zeros do not all lie on the imaginary axis but along the three two-phase lines. The model can be generalized to give rise to a tricritical point which is a new type of Yang–Lee edge singularity. Gibbs phase rule is generalized to apply to coexisting phases in the complex plane.

KEY WORDS: Yang–Lee zeros; Potts model.

1. INTRODUCTION

It was pointed out in 1952 by Yang and Lee(1) (YL) that the behavior of the zeros of the partition function in the complex plane of a field variable H is closely related to the occurrence of phase transitions. The thermodynamic properties, in the thermodynamic limit, are determined by the density of zeros of the partition function. The application of the results of YL to the study of phase transitions has been limited because of the difficulty in obtaining the distribution of these zeros. The distribution of zeros has been studied in a number of spin models.(2) For a variety of ferromagnetic models, including the Ising(1) and spherical models,(3) the zeros lie on the imaginary field axis. A rigorous proof of this result for one- and two-component ferromagnets has recently been given by Lieb and

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Sokal.[4] However, this result is not expected to be generally true and it is thus of interest to study other models.

In the thermodynamic limit the locus of YL zeros can be found by analytically continuing the magnetization M in the complex field plane and locating the branch cuts of M. The discontinuity in M across the cut is proportional to the density of zeros. The lines of zeros are thus the phase boundaries. In the one-dimensional Ising model the zeros lie on the imaginary axis beginning at $H = \pm iH_0(T)$. The edge of this gap is called the YL edge singularity and is a branch point of $M(H)$. Fisher[5] has introduced a critical exponent σ to describe the behavior of the density of zeros near this edge. In the one-dimensional Ising model $\sigma = -\frac{1}{2}$ and in mean field theories $\sigma = \frac{1}{2}$.

In this paper we study the YL zeros of the three-component ferromagnetic Potts[6] model in one dimension in the complex plane of an applied field. This model is found to have an interesting phase diagram consisting of a triple point where three phases coexist. Emerging from the triple point are three lines on which two phases coexist. These lines terminate in critical points. From a mathematical point of view the triple point corresponds to the point where three eigenvalues of the transfer matrix have the same magnitude. Such an event generally requires a special symmetry (which occurs in the Potts model). On the two phase lines two eigenvalues have the same magnitude and thus the distribution of zeros on these lines is similar to that for the one-dimensional Ising model. At the critical points the two eigenvalues become equal and the density of zeros in the thermodynamic limit diverges as in the Ising model with an exponent of $\sigma = -\frac{1}{2}$. The zeros only lie on the imaginary field axis at zero and infinite temperatures. At any finite temperature one of the two phase lines lies on the imaginary field axis and the other two lines extend into the complex plane and are complex conjugates.

By including a second complex field the region of triple points becomes a plane in the four-dimensional space of the two complex fields. This plane contains tricritical points at which the three phases simultaneously become identical. The density of zeros diverges with a $-\frac{2}{3}$ exponent as the tricritical point is approached. The tricritical point corresponds to the intersection of three planes of critical points. These considerations lead to a generalization of Gibbs phase rule to describe coexisting phases in the complex plane.

2. TRANSFER MATRIX

In the Potts model at each site of a regular lattice of N sites we place a spin which can take on q values. Nearest-neighbor like and unlike spins have an intersection energy $-\epsilon$ and 0, respectively. In the q-component model it is possible to apply $q-1$ independent "magnetic fields" to each spin. The determination of the distribution of YL zeros in this multidimensional complex space is complicated and it is desirable to simplify the problem. (We discuss the general problem in Section 8.) We choose the "magnetic field" H acting on each of the spins such that the different states of a spin labeled by $r = 1 \ldots q$ have field energies $H(q+1-2r)$ (q even) and $H[(q+1)/2 - r]$ (q odd). We treat even and odd q differently to avoid factors of $\frac{1}{2}$. There is thus a splitting of $2H$ (q even) and H (q odd) between adjacent states. With this choice for the field energies the zeros of the partition function of a single spin in the field lie on the imaginary H axis at $z^2 = e^{2H/kT} = e^{2h} = e^{2\pi i r/q}$ (q even) and $z = e^{H/kT} = e^{h} = e^{2\pi i r/q}$ (q odd) with $r = 1 \ldots q-1$.

Most of the discussion below is for the $q = 3$ component model in one dimension. We are interested in the analytical behavior of the N site partition function $Z_N(x,z)$ where $x = e^{-\epsilon/kT}$. This can be written

$$Z_N(x,z) = \mathrm{Tr}\big[T(x,z)\big]^N \tag{2.1}$$

where the transfer matrix

$$T(x,z) = \begin{bmatrix} 1 & x & x \\ x & 1 & x \\ x & x & 1 \end{bmatrix} \begin{bmatrix} z & & \\ & 1 & \\ & & \frac{1}{z} \end{bmatrix} \tag{2.2}$$

Since Z is a palindromic polynomial in z (symmetric with respect to $z \to 1/z$) with real coefficients, its roots occur in real pairs $(\xi, 1/\xi)$, in pairs of modulus one $(e^{\pm i\psi})$, or in quartets $(e^{\pm\alpha\pm i\psi})$. T has the characteristic equation,

$$f(\lambda) = \lambda^3 - 3p\lambda^2 + 3p\Delta_2\lambda - \Delta_3 = 0 \tag{2.3}$$

where $3p = z + 1 + z^{-1}$, $\Delta_2 = (1-x^2)$ and $\Delta_3 = (1-x)^2(1+2x)$. This result is generalized to q-component models in Appendix A. The partition function can be written

$$Z_N = \lambda_1^N + \lambda_2^N + \lambda_3^N \tag{2.4}$$

where the λ_i are the three roots of (2.3). It is convenient to introduce $\lambda = \Delta_3^{1/3}\sigma$, the characteristic equation becomes

$$f(\sigma) = \sigma^3 - 3\bar{p}\sigma^2 + 3\bar{p}r\sigma - 1 = 0 \tag{2.5}$$

where $\bar{p} = p/\Delta_3^{1/3}$ and $r = \Delta_2/\Delta_3^{1/3}$. The partition function is then

$$Z_N = \Delta_3^{N/3}P_N \tag{2.6}$$

with

$$P_N = \sigma_1{}^N + \sigma_2{}^N + \sigma_3{}^N \tag{2.7}$$

where the σ_i are the three roots of (2.5). For later purposes we note that

$$\begin{aligned} 3\bar{p} &= \sigma_1 + \sigma_2 + \sigma_3 \\ 3\bar{p}r &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = \sigma_1{}^{-1} + \sigma_2{}^{-1} + \sigma_3{}^{-1} \\ 1 &= \sigma_1\sigma_2\sigma_3 \end{aligned} \tag{2.8}$$

We first consider some simple cases where the roots of (2.3) are easily obtained.

(a) Zero temperature (we treat $x = e^{-\epsilon/kT}$ and z as independent variables):

$$\begin{gathered} \Delta_2 = \Delta_3 = 1, \qquad \lambda_1 = z, \qquad \lambda_2 = z^{-1}, \qquad \lambda_3 = 1 \\ Z_N = z^N + z^{-N} + 1 \end{gathered} \tag{2.9}$$

which has the roots at

$$h = \frac{2\pi i}{N}\left(l + \frac{1}{3}\right), \qquad \frac{2\pi i}{N}\left(l + \frac{2}{3}\right), \qquad l = 0, 1, \ldots, N-1$$

Thus all the YL zeros lie on the imaginary h axis.

(b) Infinite temperature:

$$\begin{gathered} \Delta_2 = \Delta_3 = 0, \qquad \lambda_1 = \lambda_2 = 0, \qquad \lambda_3 = 3p \\ Z_N = (z + z^{-1} + 1)^N \end{gathered} \tag{2.10}$$

which has roots at $h = 2\pi i/3$, $4\pi i/3$ and again the zeros lie on the imaginary axis.

(c) Triple point:

$$\begin{gathered} p = 0, \qquad z = \omega, \omega^2, \qquad \lambda_i = \Delta_3{}^{1/2}(1, \omega, \omega^2) \\ Z_N = \Delta_3{}^{N/3}(1 + \omega^N + \omega^{2N}) \end{gathered} \tag{2.11}$$

where $\omega = e^{2\pi i/3}$. Thus $Z_N = 0$ for $N \neq 3m$ and $h = 2\pi i/3, 4\pi i/3$ are zeros for all temperatures for $N \neq 3, 6, \ldots$. We will refer to this point where the three roots of (2.3) have equal magnitudes as the triple point. It will be shown below that three two-phase lines meet at this point.

(d) Zero field:

$$p = 1, \qquad h = 0, \qquad \lambda_1 = \lambda_2 = 1 - x, \qquad \lambda_3 = 1 + 2x \tag{2.12}$$

which are just the eigenvalues for the Potts model in zero field. The partition function is positive at this point.

3. RECURSION RELATIONS

In this section we show that the partition function (PF) satisfies a recursion relation. We first illustrate the method for the Ising model ($q = 2$). In this case the characteristic equation is

$$f^{(2)}(\sigma) = \sigma^2 - P_1^{(2)}\sigma + 1 = 0 \tag{3.1}$$

where $P_1^{(2)} = (z + 1/z)/\Delta_2^{1/2}$ and the PF

$$Z_N^{(2)} = \Delta_2^{N/2} P_N^{(2)} \tag{3.2}$$

We use a superscript 2 to denote the $q = 2$ case. In (3.1) $P_1^{(2)}$ is the PF of a single spin (in units of $\Delta_2^{1/2}$). Now consider $f^{(2)}(\sigma)f^{(2)}(-\sigma)$; in this product replace σ^2 by σ

$$\begin{aligned}\left[f^{(2)}(\sigma)f^{(2)}(-\sigma)\right]_{\sigma^2\to\sigma} &= \sigma^2 - \left(P_1^{(2)^2} - 2\right)\sigma + 1 \\ &= \sigma^2 - P_2^{(2)}\sigma + 1\end{aligned} \tag{3.3}$$

where $P_2^{(2)}$ is the PF for two spins. This procedure can be continued and leads to the recursion relation for the partition function

$$P_n^{(2)} = P_{n-1}^{(2)^2} - 2 \tag{3.4}$$

where $P_n \equiv P_{2^n}$. At each iteration we double the size of the lattice. Iteration procedures in which the lattice size is increased by $3, 4, \ldots$ are also possible and result from considering $f(\sigma)f(\omega\sigma)f(\omega^2\sigma)$, etc. in place of (3.3).

The recursion relation (3.4) can be written in the logistic[4] form $x_{n+1} = \lambda x_n(1 - x_n)$ with $\lambda = 4$. In order to find the zeros of P_n we must determine what values of z, i.e., P_1 are mapped into $P_n = 0$ by (3.4). Setting $P_n = 2\cos\phi_n$, Eq. (3.4) reduces to

$$\phi_n = 2\phi_{n-1} = \cdots = 2^{n-1}\phi_1 \tag{3.5}$$

and the zeros of P_n are at

$$\phi_n = 2^{n-1}\phi_1 = \left(l + \tfrac{1}{2}\right)\pi, \qquad l = 1 \ldots 2^{n-1} \tag{3.6}$$

In terms of the magnetic field the YL zeros lie on the imaginary h axis. Setting $h = i\theta$ from (3.6) we have

$$\frac{\cos\theta}{\Delta_2^{1/2}} = \cos\frac{(l + 1/2)\pi}{N} \tag{3.7}$$

[4] For a review see Ref. 7.

where we have replaced 2^{n-1} by N. The density of zeros[1] per spin is given by

$$g(\theta)=\frac{1}{\pi}\left|\frac{\partial\phi_1}{\partial\theta}\right|=\frac{1}{\pi}\,\frac{\sin\theta}{\left(\Delta_2-\cos^2\theta\right)^{1/2}} \tag{3.8}$$

and thus diverges with an exponent $\sigma=-\frac{1}{2}$ at the YL edge $\cos^2\theta=\Delta_2$.

The above procedure can also be applied to the Potts model. In the three-component case the characteristic function Eq. (2.5) $f(\sigma)=\sigma^3-P_1\sigma^2+Q_1\sigma-1$, where $P_1=3\bar{p}$ and $Q_1=3\bar{p}r$. P_1 is the partition function of a single spin (in unity of $\Delta_3{}^{1/3}$). Following the same procedure as in (3.3) we obtain the recursion relations

$$\begin{aligned} P_n &= P_{n-1}^2-2Q_{n-1}\\ Q_n &= Q_{n-1}^2-2P_{n-1} \end{aligned} \tag{3.9}$$

which now involve two parameters. Q_n may be shown to be the partition function of a reciprocal Potts model. Thus the function $f(\sigma)$, Eq. (2.5), is preserved under the transformations $\bar{p}\to\bar{p}r$, $r\to r^{-1}$, $\sigma\to\sigma^{-1}$. Then

$$\begin{aligned} Q_n(\bar{p},r) &= P_n(\bar{p}r,r^{-1})\\ P_n(\bar{p},r) &= Q_n(\bar{p}r,r^{-1}) \end{aligned} \tag{3.10}$$

so that Q_n is the PF of the same model with parameters $\bar{p}r$, r^{-1}. Any root of $P_n(\bar{p},r)$, say, $\bar{p}=p_i(r)$, gives a root of $Q_n(\bar{p},r)$, $\bar{p}=(1/r)p_i(1/r)$.

The recursion relations (3.9) can be used to analyze the properties of the Potts model PF. However, the properties of the recursion relations are most easily obtained from the roots σ_i of (2.5) because $P_n=\sigma_1{}^{2^{n-1}}+\sigma_2{}^{2^{n-1}}+\sigma_3{}^{2^{n-1}}$, $Q_n=\sigma_1{}^{-2^{n-1}}+\sigma_2{}^{-2^{n-1}}+\sigma_3{}^{-2^{n-1}}$. At a triple point where we have three roots of unit magnitude the recursion relations map into themselves. At any other point the recursion relations eventually map to infinity because there are one or more roots of magnitude greater than 1. If there are two roots $\sigma_1=\mathrm{R}e^{i(\phi+\psi)}$, $\sigma_2=\mathrm{R}e^{i(\phi-\psi)}$ with $R>1$ then after a number of iterations these two roots dominate P_n and $Q_n\simeq\sigma_3{}^{-2^{n-1}}=R^{2^n}e^{i\phi 2^n}$. The recursion relations then become of the Ising form (3.5) and the angles ϕ,ψ double at each iteration. Some further properties of the recursion relations are given in Section 8. We now turn to a study of the roots of (2.3).

4. ROOTS FOR REAL p

We consider the roots of (2.3) for p real and write it as

$$3p=\frac{\lambda^3-\Delta_3}{\lambda(\lambda-\Delta_2)} \tag{4.1}$$

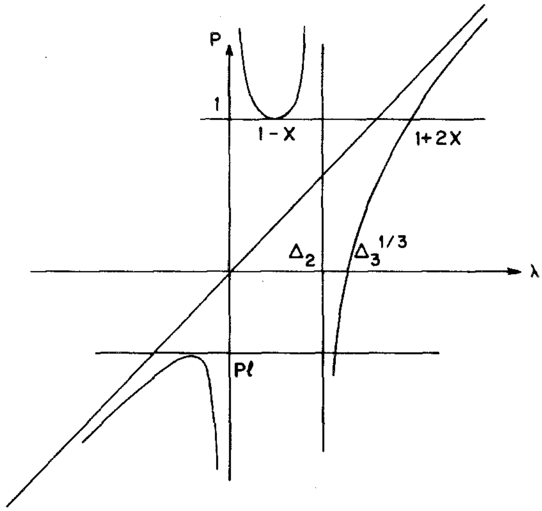

Fig. 1. Sketch of Eq. (4.1).

This function is sketched in Fig. 1. At $p = 1$ there is a double root $\lambda = 1 - x$ and a single one $\lambda = 1 + 2x$ and for $p > 1$ there are three positive roots and $Z_N > 0$. For $p < p_l$ we have one positive and two negative roots and again it can be shown that no zeros of the PF occur. The two negative roots coalesce at p_l which is determined by eliminating λ between $f(\lambda) = 0$ and $f'(\lambda) =$, i.e., the vanishing of the discriminant of $f(\lambda)$ (see Section 5).

For $p_l < p < 1$ we have $z = e^{i\theta}$, $p = \frac{1}{3}(1 + 2\cos\theta)$ with $\pi > \theta > 0$. We have one positive root λ_3 with $\Delta_2 < \lambda_3 < 1 + 2x$ and two complex conjugate roots $\lambda_{1,2} = \mathrm{Re}^{\pm i\psi}$. These satisfy the equations

$$\begin{aligned} 3p &= \lambda_3 + 2R\cos\psi \\ 3p\Delta_2 &= R^2 + 2\lambda_3 R\cos\psi \\ \Delta_3 &= \lambda_3 R^2 \end{aligned} \tag{4.2}$$

For $p = 0$ $\lambda_3 = R = \Delta_3{}^{1/3}$, $\theta = \psi = \pm 2\pi/3$. This is an invariant point of the θ, ψ mapping. Choose $0 < \psi < \pi$ and then

$$\begin{aligned} &1 > p > 0, \qquad 0 < \psi < \frac{2\pi}{3}, \qquad R < \lambda_3 \\ &p_l < p < 0, \qquad \pi > \psi > \frac{2\pi}{3}, \qquad R > \lambda_3 \end{aligned} \tag{4.3}$$

We are now in a position to consider the PF

$$Z_N = \lambda_3{}^N + 2R^N \cos N\psi \tag{4.4}$$

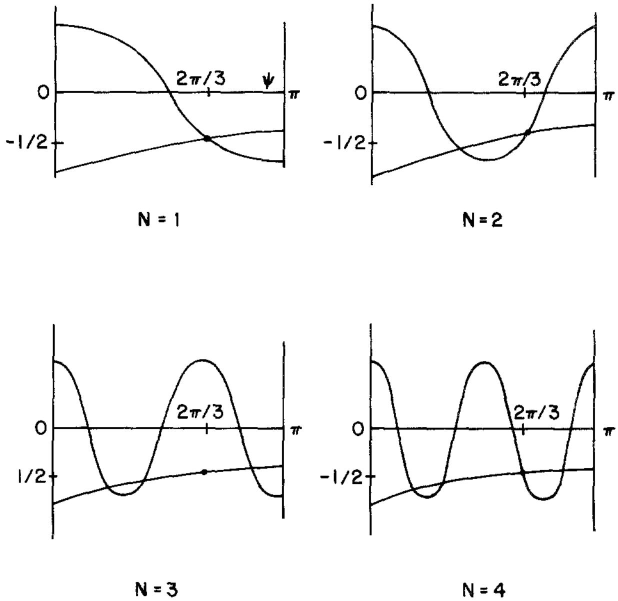

Fig. 2. Sketch of $\cos N\psi$ and $-(1/2)(\lambda_3/R)^N$ versus ψ. For $\psi = 2\pi/3$ the latter passes through $-1/2$.

From (4.2) λ_3, R, and p can be considered to be functions of ψ, the intersections of $\cos N\psi$ and $-(1/2)(\lambda_3/R)^N$ are sketched in Fig. 2 for small N. The line $-(1/2)(\lambda_3/R)^N$ always passes through $-1/2$ for $\psi = 2\pi/3$. For $x = 0$, this line is horizontal and there are N intersections corresponding to $Z_N = 0$. The intercept of this line at $\psi = 0$ is $-(1/2)[(1 + 2x)^N/(1 - x)]$. As x increases, this line approaches the ψ axis for $\psi > 2\pi/3$ and moves away from this axis for $\psi < 2\pi/3$. Consider the cases $N \geqslant 3$ in Fig. 2. As x increases the two roots near π/N coalesce to become a double root and then move into the complex p plane as x is further increased. For $N = 3, 4, 5$ only one such double root occurs, but for $N \geqslant 6$ there can be two such double roots, for $N \geqslant 9$ there can be three such double roots, etc. The values of ψ near where this behavior occurs are $\psi = \pi/N, 3\pi/N, \ldots, (2r-1)\pi/N$, $r = 1 \ldots [N/3]$, where $[N/3]$ is the integer part of $N/3$. For $-1/3 < p < 1$, z lies on the unit circle and when p becomes complex z moves off the unit circle. Consider some PF $Z_N(x,z)$. At $x = 0$ all the zeros lie on the unit circle and as x increases the first pair of zeros, say, z_1, z_2 and z_1^*, z_2^*, move toward each other and become double zeros. As x further increases they split and move away from the unit circle perpendicularly forming a quartet $[re^{\pm i\theta}, (1/r)e^{\pm i\theta}]$. This process is then repeated by the next pair of zeros on the unit circle and the process is repeated at increasing θ until the last pair before $\theta = 2\pi/3$ coalesce and the process stops. As N increases the process occurs at smaller values of x and in the limit $N \to \infty$, $x > 0$ we obtain three branches of zeros which meet at the triple point $p = 0$, $\theta = \psi = \pm 2\pi/3$. In terms of the magnetic field for x sufficiently small all the YL zeros lie on the imaginary h axis. As x increases they split off in pairs giving rise to zeros at $\pm h_i \pm i\theta_i$ where h_i is real. In the $N \to \infty$ limit in the upper half h plane we have three lines of zeros meeting at $\theta = 2\pi/3$. The zeros on the imaginary h axis lie in the region $2\pi/3 < \theta < \theta_l$, where $2\cos\theta_l + 1 = p_l$. We now analyze in more detail the positions of the zeros for complex p.

5. CRITICAL POINTS

The critical points where the lines of zeros terminate (for $N \to \infty$) occur at those values of p at which we have two equal roots of the characteristic equation (2.3). The equation for p is obtained by elimination of λ between $f(\lambda) = 0$ and $f'(\lambda) = 0$, i.e., the vanishing of the discriminant of $f(\lambda)$. This is easily shown to be proportional to

$$D(p) = (\Delta_3 - p^2\Delta_2) - 4(p\Delta_2 - p^2)(p\Delta_3 - p^2\Delta_2^2) \tag{5.1}$$

which is a quartic in p. We already know one root corresponding to the zero field case $p = 1$, $\lambda = 1 - x, 1 - x, 1 + 2x$. The other three roots correspond to the critical points and there is one real, negative root p_l and two complex conjugate roots. These are given in Appendix B.

At low temperatures ($x \ll 1$) the critical points are at

$$p_l = -\frac{1}{3} + \frac{x^2}{3} - \frac{x^3}{3}, \qquad 1 - \frac{3x^2}{2}\left(1 \pm i/\sqrt{3}\right) \tag{5.2}$$

At high temperatures ($1 - x \ll 1$) the zeros move toward $p = 0$ and the critical points are at

$$p_l = \left[\tfrac{3}{4}(1 - x^2)\right]^{1/3}(-1, e^{\pm \pi i/3}) \tag{5.3}$$

Thus the phase diagram (lines of zeros of the PF) consists of three lines beginning at the triple point $p = 0$ and terminating at the above three critical points. On these lines two phases coexist, while at the triple point three phases coexist. At the critical points where two roots of the characteristic equation become equal the density of zeros (for an infinite system) will diverge with an exponent of $-1/2$ as in the Ising case Eq. (3.8). At the triple point the density of zeros is finite.

In the h plane the triple points lie at $h = \pm 2\pi i/3$. The critical points at low and high temperatures may be found from (5.2) and (5.3):

$$\begin{aligned} h &= i(\pi - x), \qquad 3^{3/4} x e^{\pi i/2(1\pm 1/6)}, \qquad x \ll 1 \\ h - 2\pi i/3 &= i\sqrt{3}\left[3/4(1-x)^2\right]^{1/3}(1, e^{-\pi i/2(1\mp 2/3)}), \qquad 1 - x \ll 1 \end{aligned} \tag{5.4}$$

There is another set of critical points in the lower half h plane obtained by taking the complex conjugate of the above results.

6. DISTRIBUTION OF ZEROS

In the thermodynamic limit the locus of zeros of the PF can be found by analytically continuing the magnetization in the complex field plane and locating the branch cuts of M. The discontinuity in M across the cut is proportional to the density of zeros. In order for the magnetization to show a cut it is necessary that the two large eigenvalues have equal magnitude. Thus we can determine the locus of zeros as those lines in the complex p plane where two roots of the characteristic equation have the same magnitude. In this plane there are three such lines beginning at the triple point $p = 0$ and terminating at the critical points determined in the previous section.

(a) On one of the two phase lines p is real $p_l < p < 0$ and the roots have the form

$$\sigma_1 = \sigma_2^* = \bar{R} e^{i\psi}, \qquad \sigma_3 = \bar{R}^{-2} \tag{6.1}$$

Substituting in (2.8) and eliminating $\bar{p}$ gives

$$2\cos\psi = \frac{\bar{R}^4 - r}{r\bar{R}^3 - \bar{R}} \tag{6.2}$$

The curve of $\bar{R}$ versus ψ has been determined numerically and is shown in Fig. 3.

(b) The other two phase lines are complex conjugates. On the line where $|\sigma_1| = |\sigma_3|$ the roots can be written in the form

$$\sigma_1 = \bar{R} e^{i(\phi+\psi)}, \qquad \sigma_2 = \bar{R} e^{i(\phi-\psi)}, \qquad \sigma_3 = \bar{R}^{-2} e^{-2i\phi} \tag{6.3}$$

Substituting these results in (2.8) and eliminating $\bar{p}$ the real and imaginary parts can be written

$$\begin{aligned} 2\cos\phi &= \left[\frac{(r\bar{R}^2 + 1)(\bar{R}^4 - r)}{\bar{R}^2(\bar{R}^2 - r^2)}\right]^{1/2} \\ \cos\psi &= \frac{\bar{R}^4 + r}{\bar{R}(\bar{R}^2 r + 1)}\cos\phi \end{aligned} \tag{6.4}$$

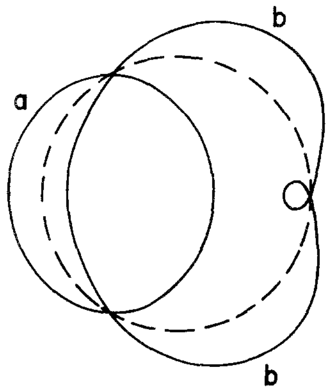

Fig. 3. Pairs of roots of equal magnitude for $r = 0.5$. The unit circle is shown dotted. The YL zeros (in the thermodynamic limit) correspond to the regions outside the unit circle. (a) The line $\bar{R}e^{i\psi}$ on which $|\sigma_1| = |\sigma_2|$; (b) the line $\bar{R}e^{i(\phi+\psi)}$ on which $|\sigma_1| = |\sigma_3|$ and its complex conjugate b^* on which $|\sigma_2| = |\sigma_3|$.

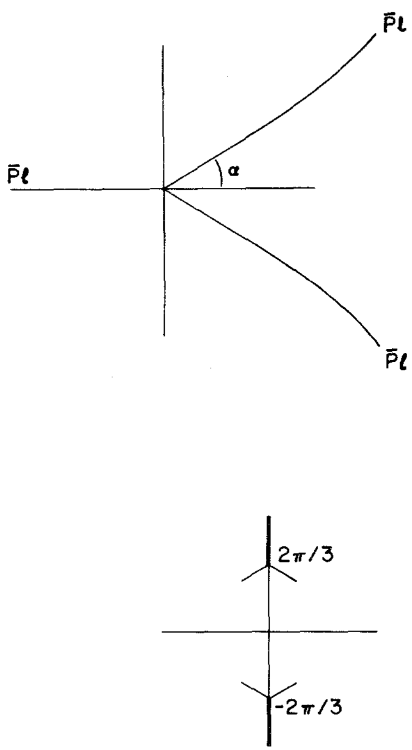

Fig. 4. (a) Phase diagram in the p plane for $r = 0.5$. (b) Phase diagram in the H/kT plane.

which are again suitable for numerical evaluation. The results are shown in Fig. 3.

From the values of the roots we can determine $\bar{p}$ and the phase diagram in the $\bar{p}$ plane (Fig. 4a) and the h plane (Fig. 4b).

Close to the triple point the two complex two-phase lines each make an angle α with the real positive p axis given by

$$\tan\alpha = \sqrt{3}\,\frac{1-r}{1+r} \tag{6.5}$$

as $T \to 0$ $r \to 1$ and α vanishes. In the h plane the two complex critical lines in the upper half-plane make the same angle α with the negative imaginary axis.

7. MAGNETIZATION

In the thermodynamic limit the locus of YL zeros corresponds to branch cuts in the magnetization M. The magnitude of the discontinuity in M across the cut is proportional to the density of zeros $g(h)$:

$$g(h) = \frac{1}{2\pi}\,|M(h_+) - M(h_-)| \tag{7.1}$$

where $h_\pm$ refers to the two sides of the cut. By differentiation of (2.7) and (2.8) with respect to h,

$$\begin{aligned} M &= \frac{1}{N}\,\frac{\partial \ln Z}{\partial h} \\ &= \frac{2\sinh h}{\Delta_3^{1/3} P_N}\left[\sigma_1^N f_1 + \sigma_2^N f_2 + \sigma_3^N f_3\right] \end{aligned} \tag{7.2}$$

where

$$f_1 = \frac{\sigma_1 - r}{(\sigma_1 - \sigma_2)(\sigma_1 - \sigma_3)} \tag{7.3}$$

and $f_{2,3}$ are obtained by cyclic permutations of the subscripts. In the thermodynamic limit Z is determined by the eigenvalue of greatest magnitude and in order for the magnetization to show a cut it is necessary that at least two eigenvalues have equal magnitudes on the cut. These eigenvalues have been denoted by σ_1 and σ_2 in Eq. (6.1) and then the density of YL zeros is given in terms of the eigenvalues by[5]

$$g = \left|\frac{\sinh h}{\pi \Delta_3^{1/3}}\,(f_1 - f_2)\right| \tag{7.4}$$

This formula can be used to evaluate the density of zeros. We have already

[5] The same formula can be obtained by noting that the density of zeros along the two-phase line (6.1) is uniform in ψ and equal to $1/\pi$ for large N. Thus $g(h) = (1/\pi)|d\psi/dh|$.

noted that at the critical points the density diverges as in Ising case Eq. (3.8) with an exponent of $-1/2$. Close to the triple point it is found that

$$g = \frac{1+r}{2\pi\Delta_3^{1/3}}, \qquad \frac{\left(1-r+r^2\right)^{1/2}}{2\pi\Delta_3^{1/3}} \tag{7.5}$$

where the two results refer to the branches (6.3) and (6.4), respectively.

8. DISCUSSION

In this paper we have discussed the distribution of YL zeros in the one-dimensional, three-component Potts model. The new feature in this model, not present in the Ising model, is the possibility of three roots of equal magnitude which leads to the appearance of a triple point in the phase diagram. It is clear that in higher-component Potts models more complicated multiple points will occur. The YL zeros at zero temperature lie on the imaginary field axis but as the temperature is raised they split off in pairs from this axis forming, in the thermodynamic limit, the phase diagram shown in Fig. 4.

We have discussed in detail the case where the applied field splits the spin states equally in energy. In general two independent fields, say, H_A and H_B, can be applied to the three-component Potts model. In this case a new feature can appear in the phase diagram, i.e., a tricritical point where the three phases simultaneously become identical. For two independent fields Eq. (2.5) generalizes to

$$f(\sigma) = \sigma^3 - 3p_1\sigma^2 + 3p_2 r\sigma - 1 = 0 \tag{8.1}$$

where

$$\begin{aligned} p_1 &= \frac{1}{3\Delta_3^{1/3}}\left(z_A + z_B + (z_A z_B)^{-1}\right) \\ p_2 &= \frac{1}{3\Delta_3^{1/3}}\left(z_A^{-1} + z_B^{-1} + z_A z_B\right) \end{aligned} \tag{8.2}$$

where $z_{A,B} = e^{H_{A,B}/kT}$. The energies of a spin in the fields are H_A, H_B, $-H_A - H_B$. H_A and H_B are complex variables so that the phase space is four dimensional.

The region of triple points is determined by requiring that (8.1) have three roots of unit magnitude. If we write these in the form $e^{i\psi_1}$, $e^{i\psi_2}$, and $e^{-i(\psi_1+\psi_2)}$, then

$$3p_1 = 3p_2^* r = e^{i\psi_1} + e^{i\psi_2} + e^{-i(\psi_1+\psi_2)} \tag{8.3}$$

The triple points thus lie on a two-dimensional surface (at constant T). Equation (8.3) can be solved for z_A, z_B, and $(z_A z_B)^{-1}$ and they are the three

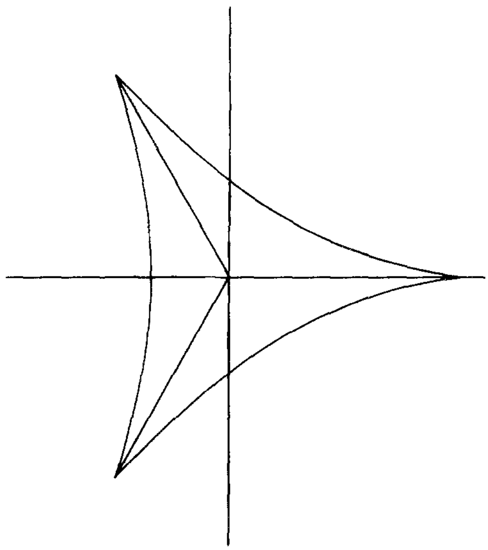

Fig. 5. Deltoid region of triple points in the p_1 plane. The tricritical points are at the vertices $1, \omega, \omega^2$.

solutions of

$$z^3 - 3p_1\Delta_3{}^{1/3}z^2 + \frac{3p_1^*}{r}\Delta_3{}^{1/3}z - 1 = 0 \tag{8.4}$$

In the p_1 space the surface of triple points (8.3) is a deltoid shown in Fig. 5. On the edges of the deltoid we have $\psi_1 = \psi_2$ so that on these lines two phases become identical. At the corners of the deltoid the three roots of (8.1) are equal and are $\sigma_i = 1$, ω, or ω^2. These points are thus tricritical points. We also note that the region of triple points is preserved under the map (3.9). The tricritical point $p_1 = 1$ is an unstable fixed point of the map (3.9) and the other two $p_1 = \omega, \omega^2$ are an unstable two-cycle.

The behavior of the density of zeros near a tricritical point is of interest. If we set $3\bar{p}_1 = 3 - \delta_1$, $3r\bar{p}_2 = 3 + \delta_2$ where δ_1 and δ_2 are small it is easily shown that the roots of (8.1) are

$$\sigma_1 \approx 1 + \delta^{1/3}, \qquad \sigma_2 \approx 1 + \omega\delta^{1/3}, \qquad \sigma_3 \approx 1 + \omega^2\delta^{1/3} \tag{8.5}$$

where $\delta = \delta_1 + \delta_2$. From (7.2) we conclude that the density of zeros diverges as $\delta^{-2/3}$ as we approach the tricritical point leading to an exponent $\sigma = -2/3$ for a tricritical point. This assumes that the direction of approach to the tricritical point is not tangential to any of the lines of critical points meeting there. On these lines the density of zeros diverges with the Ising exponent $\sigma = -1/2$. A crossover function can be constructed to describe this behavior.

We have found that there are interesting multiple and multicritical points in the complex plane of the field variables. This suggests a generalization of Gibbs phase rule. Suppose we have q components and m coexisting phases. The intensive thermodynamic variables are T and $(q-1)$ complex field variables giving a total $2q-1$ intensive variables ($2q$ if we include the pressure). For real fields the condition for two coexisting phases is that the free energies of the phases be equal. When the intensive variables are complex, the PF and the free energy are complex and it is necessary to generalize this condition. In this paper we have found that, in the thermodynamic limit, two coexisting phases require that the two largest eigenvalues of the transfer matrix have the same magnitude; three coexisting phases require that the three largest eigenvalues have the same magnitude, etc. Thus the PF's of coexisting phases have the same magnitude but different phase angles. This suggests that the condition for coexisting phases in the complex plane requires that the real part of the free energies be equal and that the phase angle of the PF (imaginary part of the free energy) can be arbitrary. Thus m coexisting phases lead to $m-1$ conditions and the number of thermodynamic degrees of freedom is $f = 2q - m$, ($2q - m + 1$ if the pressure is included). Thus for $q = 3$ the triple points ($m = 3$) occupy a three-dimensional region (including T) and double points ($m = 2$) a four-dimensional region. At a critical point an extra condition arises and $f = 3$, and at a tricritical point two extra conditions arise and $f = 1$.

APPENDIX A

We derive the characteristic equation for the one-dimensional q-component Potts model in the case of a single field which splits the states equally. If no external field is present the characteristic function is

$$\lambda^q + \binom{q}{1}\Delta_1(x)\lambda^{q-1} + \binom{q}{2}\Delta_2(x)\lambda^{q-2} \cdots \mp \Delta_q(x) \tag{A1}$$

where $\Delta_q(x) = (1-x)^{q-1}[1 + (q-1)x]$ and the upper and lower signs are for q odd and even, respectively. Similarly, if $x = 0$ and the field is present ($z = e^{H/kT}$) the characteristic function is

$$\begin{aligned} g_q(\lambda) &= (\lambda - z^{-(q-1)/2}) \ldots (\lambda - z^{-1})(\lambda - 1)(\lambda - z) \\ &\quad \ldots (\lambda - z^{(q-1)/2}), && q \text{ odd} \\ &= (\lambda - z^{-(q-1)}) \ldots (\lambda - z^{-1})(\lambda - z) \ldots (\lambda - z^{q-1}), && q \text{ even} \end{aligned} \tag{A2}$$

In either case we obtain an equation like

$$g_q(\lambda) = \lambda^q - \phi_1^{(q)}(z)\lambda^{q-1} + \phi_2^{(q)}(z)\lambda^{q-2} \cdots \mp 1 \tag{A3}$$

with $\phi_k^{(q)}(1) = \binom{q}{k}$, $\phi_k(z) = \phi_{q-k}(z)$.

When $x \neq 0$, $z \neq 1$ the characteristic function is

$$f(\lambda) = \lambda^q - \phi_1^{(q)}(z)\Delta_1(x)\lambda^{q-1} + \phi_2^{(q)}(z)\Delta_2(x)\lambda^{q-2} \cdots \mp \Delta_q(x) \tag{A4}$$

We now study the properties of $\phi_k^{(q)}(z)$. First consider q even and change λ to $-\lambda$ for convenience. Then

$$g_q(-\lambda) = \left(1 + \frac{\lambda}{z^{q-1}}\right)\left(1 + \frac{\lambda}{z^{q-3}}\right) \cdots \left(1 + \frac{\lambda}{z^{q-1}}\right)$$

This function satisfies a functional equation

$$(1 + \lambda z^{-(q-1)})\, g_q(-\lambda z^2) = (1 + \lambda z^{q+1})\, g_q(-\lambda) \tag{A5}$$

which can be used to determine the series (A3). Substituting this series in (A5) and equating coefficients of λ on both sides gives

$$\phi_{k+1}^{(q)}(z) = \frac{z^{q-k} - z^{-(q-k)}}{z^{k+1} - z^{-(k+1)}}\, \phi_k^{(q)}(z) = \frac{\sinh(q-k)\beta H}{\sinh(k+1)\beta H}\, \phi_k^{(q)}(z) \tag{A6}$$

which gives an iterative procedure for determining the ϕ_k. For q odd the same result holds if we replace z by $z^{1/2}$.

APPENDIX B

The characteristic equation (2.5) has double roots whenever its discriminant $D(p)$, a quartic, vanishes; there are four such roots. Instead of analyzing $D(p)$ it is simpler to study $f(\sigma)$. Let τ be the double root; then $\sigma_1 = \sigma_2 = \tau$ and $\sigma_3 = \tau^{-2}$. The two equations (2.8) involving p as a symmetric function of the roots are

$$\begin{aligned} 3\bar{p} &= 2\tau + \tau^{-2} \\ 3r\bar{p} &= 2\tau^{-1} + \tau^2 \end{aligned} \tag{B1}$$

A quartic $g(\tau)$ can be obtained by eliminating $\bar{p}$ between the two equations above

$$g(\tau) = \tau^4 - 2r\tau^3 + 2\tau - r \tag{B2}$$

It can be shown that $g(\tau)$ has only one root inside the unit circle for $0 < r < 1$. Bowman[8] considers the equation

$$sn(2u, \gamma) = sn(K(\gamma) - u, \gamma) \tag{B3}$$

where γ is the modulus and $4K(\gamma)$ is the real period of the elliptic function $sn(u,\gamma)$. From Eq. (B3) a quartic in $s = sn(u,\gamma)$ can be derived:

$$\gamma^2 s^4 - 2\gamma^2 s^3 + 2s - 1 = 0 \tag{B4}$$

Bowman finds the following amplitudes:

$$u = \tfrac{1}{3}K(\gamma), \qquad 3K(\gamma) + \tfrac{2}{3}iK'(\gamma), \qquad \tfrac{1}{3}K(\gamma) \pm \tfrac{2}{3}iK'(\gamma) \tag{B5}$$

thus solving Eq. (B3). If in Eq. (B4) we let $\tau = rs$; $r^3 = \gamma^2$ then Eqs. (B2) and (B4) become identical. An irreducible set of roots is

$$\begin{aligned} \tau_1 &= rsn\left(\tfrac{1}{3}K, \gamma\right) \\ \tau_2 &= rsn\left(3K + \tfrac{2}{3}iK', \gamma\right) \\ \tau_{3,4} &= rsn\left(\tfrac{1}{3}K \pm \tfrac{2}{3}iK', \gamma\right) \end{aligned} \tag{B6}$$

$$\{\tau_1 > 0, \tau_2 < 0, \tau_3 = \tau_4^*\}, \qquad \{|\tau_1| < 1 < |\tau_i|, i = 2,3,4\}$$

Another quartic $h(\pi)$ can be constructed from $g(\tau)$ by forming $\pi^4 g(1/\sqrt{\pi})g(-1/\sqrt{\pi}) = h(\pi)$. Clearly the roots of $h(\pi)$ are τ_1^{-2}, τ_2^{-2}, τ_3^{-2}, τ_4^{-2}. The reciprocal transformation

$$\pi = \frac{r - \bar{p}}{1 - r^2\bar{p}} \tag{B7}$$

applied to $h(\pi) = 0$ implies $D(p) = 0$. The roots of the discriminant are

$$r^2\bar{p}_j = dn^2(u_j, \gamma)/cn^2(u_j, \gamma) = \gamma^2 sn^2(u_j + K + iK', \gamma)$$

An irreducible set of roots of the discriminant $D(p)$ is

$$\begin{aligned} \bar{p}_1 &= rsn^2\left(\tfrac{2}{3}K + iK', \gamma\right), & \bar{p}_3 &= rsn^2\left(\tfrac{2}{3}K + \tfrac{1}{3}iK', \gamma\right) \\ \bar{p}_2 &= rsn^2\left(\tfrac{1}{3}iK', \gamma\right), & \bar{p}_4 &= rsn^2\left(\tfrac{2}{3}K - \tfrac{1}{3}iK', \gamma\right) \end{aligned} \tag{B8}$$

$$\{\bar{p}_1 > 0, \bar{p}_2 < 0, \bar{p}_3 = \bar{p}_4^*\}$$

The same result can be obtained by noting that the discriminant $D(p)$ occurs as the denominator of the expansion of $sn(3u,\gamma), cn(3u,\gamma), dn(3u,\gamma)$ in powers of $s = sn(u,\gamma)$ (see Ref. 8).

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